

Noether symmetries and duality transformations in cosmology

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We discuss the relation between Noether (point) symmetries and discrete symmetries for a class of minisuperspace cosmological models. We show that when a Noether symmetry exists for the gravitational Lagrangian then there exists a coordinate system in which a reversal symmetry exists. Moreover as far as concerns the scale-factor duality symmetry of the dilaton field, we show that it is related to the existence of a Noether symmetry for the field equations, and the reversal symmetry in the normal coordinates of the symmetry vector becomes scale-factor duality symmetry in the original coordinates. In particular the same point symmetry as also the same reversal symmetry exists for the Brans-Dicke- scalar field with linear potential while now the discrete symmetry in the original coordinates of the system depends on the Brans-Dicke parameter and it is a scale-factor duality when $\omega_{BD} = 1$. Furthermore, in the context of the O'Hanlon theory for $f(R)$ -gravity, it is possible to show how a duality transformation in the minisuperspace can be used to relate different gravitational models.

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1. INTRODUCTION

The necessity to explain the new phenomena, which have been discovered from the analysis of the recent cosmological data [1–6], has led to gravitational theories which extend Einstein's General Relativity (GR). The modification\extension of the Einstein-Hilbert action which keeps the linearity of the gravitational action is the introduction of the cosmological constant, Λ , which is the simplest candidate to explain the late-time acceleration of the universe.

Inspired from the field theory, new scalar fields, minimally or nonminimally coupled to gravity, either higher-order curvature invariants have been introduced into the gravitational action, for instance see [7–14]. The main issue of modified\extended theories of gravity is that the new components in the gravitational field equations, which follow from the new terms in the gravitational action, change the dynamics of the field equations such as the solutions of the latter to describe the observable phenomena. The extra terms arising from the modification of gravitational action can be seen as the components of a sort of effective geometric dark energy momentum tensor [15]. However, such terms can be related to properties coming from fundamental physics.

In particular, an important property of the two-dimensional conformal field theory and, consequently of the string theory, is the duality symmetry which has important consequences in cosmology (for reviews see [16, 17]). As duality symmetry is characterized by the invariance of the action integral, therefore the corresponding Euler-Lagrange equations, under this transformation, remain the same. Usually when the "radius" \mathcal{R} of the geometry changes such as $\mathcal{R} \rightarrow \mathcal{R}^{-1}$. Duality transformation is a discrete transformation and an isometry should exist for the underlying manifold [19, 20]. However isometries form a subalgebra on the Homothetic algebra of a manifold, while the latter algebra is related to the group of local invariant transformations which transforms the action integral in such a way that the Euler-Lagrange equations are invariant. These are the so called Noether (point) symmetries [21].

Furthermore, the dilaton scalar field model in a Friedmann-Lemaître-Robertson-Walker spacetime (FLRW) admits a scale-factor duality symmetry [22], that is, when the scale-factor changes such as $a \rightarrow a^{-1}$, and the dilaton field is shifted with a given formula, the gravitational action remains invariant. As we will see below, in the minisuperspace approach, the dilaton action admits a Noether conservation law, while the Noether symmetry and the conservation law

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are also invariant under duality transformations. However Noether symmetries provide point transformations where the field equations are invariant, despite of the fact that the duality transformations are discrete transformations.

Mathematically the two identities are different, however as we will see, for a class of cosmological models, the existence of a Noether conservation law indicates the existence of a discrete transformation, i.e. of a discrete symmetry, (not necessary that of scale-factor duality transformation).

This work focuses on the relation between discrete transformations following from the existence of Noether symmetries in cosmological Lagrangians. As a byproduct, we will show that duality transformations are a particular case of these discrete transformations.

The plan of the paper is the following. In Section 2, we briefly discuss the duality transformation of the sigma model and the scale-factor duality of the dilaton field. Using previous results for Noether symmetries in minisuperspace, in Section 3, we discuss the existence of Noether conservation laws, i.e. local transformations, with the existence of discrete transformations in the normal coordinates of the symmetry vector. In particular, we show that the existence of the Homothetic vector field, not necessary an isometry vector in the minisuperspace, generates at least a Noether symmetry for the corresponding Lagrangian. It is equivalent to the existence of a discrete transformation which leaves invariant the dynamical system. This approach is used in Section 4, where we show that the Brans-Dicke scalar field, with linear potential, admits scale-factor duality symmetry for the Brans-Dicke parameter $\omega_{BD} = 1$. The Brans-Dicke field corresponds to the dilaton field also for models with arbitrary parameter ω_{BD} admitting the same number of Noether symmetries. However the transformation from the original coordinates to the normal coordinates depends on ω_{BD} , and the reverse symmetry, in the normal coordinates, becomes scale-factor duality in the original coordinates if and only if $\omega_{BD} = 1$. Moreover, another kind of discrete transformation exists for other values of the Brans-Dicke parameter. Finally, we consider the O'Hanlon theory, which is equivalent to the $f(R)$ -gravity. We find a family of potentials that can be related each other under a discrete transformation which is a duality transformation in the minisuperspace. In Section 5, we draw conclusions.

2. STRING DUALITY AND SCALE FACTOR DUALITY

Following Buscher [19, 20] we consider the general dualizable bosonic non-linear sigma model on a D -dimensional manifold \mathcal{M} of Lorentzian signature with a dilaton field ϕ coupled to the curvature scalar of the two-dimensional metric tensor $\gamma_{\mu\nu}$. The action integral of the latter model can be written in the following form

$$S = \frac{1}{4\pi a'} \int d^2\xi \left(\sqrt{\gamma} \gamma^{\mu\nu} g_{ab} \partial_\mu x^\alpha \partial_\nu x^b + \varepsilon^{\mu\nu} h_{ab} \partial_\mu x^\alpha \partial_\nu x^b + a' \sqrt{\gamma} R^{(2)} \phi(x^\gamma) \right), \quad (1)$$

where g_{ab} is the metric tensor of the target space, h_{ab} is the torsion, and a' is the inverse of the string tension. We assume that the manifold M , admits an isometry vector field, specifically a translation symmetry, that is the \mathcal{M} manifold can seen as $1 + (D - 1)$ space.

Hence, the dualized model has the following action integral

$$\bar{S} = \frac{1}{4\pi a'} \int d^2\xi \left(\sqrt{\gamma} \gamma^{\mu\nu} \bar{g}_{ab} \partial_\mu \bar{x}^\alpha \partial_\nu \bar{x}^b + \varepsilon^{\mu\nu} \bar{h}_{ab} \partial_\mu \bar{x}^\alpha \partial_\nu \bar{x}^b + a' \sqrt{\gamma} R^{(2)} \phi(x^\gamma) \right), \quad (2)$$

where the new metric tensor \bar{g}_{ab} , and the new torsion \bar{h}_{ab} are related with that of (1) as follows

$$\bar{g}_{00} = (g_{00})^{-1}, \quad \bar{g}_{0i} = h_{0i} (g_{00})^{-1}, \quad \bar{g}_{ij} = g_{ij} - (g_{0i} g_{0j} - h_{0i} h_{0j}) (g_{00})^{-1}, \quad (3)$$

$$\bar{h}_{0i} = -h_{i0} = g_{0i} (g_{00})^{-1}, \quad \bar{h}_{ij} = h_{ij} + (g_{0i} h_{0j} - h_{0i} g_{0j}) (g_{00})^{-1}, \quad (4)$$

in which $i = 1, 2, 3 \dots D$.

The two action integrals (1) and (2), through the variational principle, are equivalent at classical level. However, in order to be equivalent at the quantum level, the action integral (1) has to be conformally invariant. Conformal invariance of (1), at the one-loop level, gives that the dilaton field ϕ has to satisfy the following conditions [22]

$$\frac{1}{a'} \frac{D-26}{48\pi^2} + \frac{1}{16\pi^2} \left(4(\nabla\phi)^2 - 4\nabla^2\phi - R + \frac{1}{12}H^2 \right) = 0, \quad (5)$$

$$R_{ab} - \frac{1}{4} H_a^{cd} H_{bcd} + 2\nabla_a \nabla_b \phi = 0, \quad (6)$$

$$\nabla_c H_{ab}^c - 2(\nabla_c \phi) H_{ab}^c = 0, \quad (7)$$

where R_{ab} is the Ricci tensor related to the metric tensor g_{ab} , R is the Ricci scalar, $H_{abc} = 3\nabla_{[a} h_{bc]}$ is the antisymmetric tensor strength, and $H^2 = H_{abc} H^{abc}$. Therefore the dualized model (2) is conformally invariant if dilaton field shifts as

$$\bar{\phi} = \phi - \frac{1}{2} \ln(g_{00}). \quad (8)$$

The set of discrete transformations which relates the two models (1), (2) form the duality transformations of the theory. Furthermore, the transformation is an element of the discrete symmetry group $O(d, d)$ [24].

2.1. Scale-factor duality

The equations that the dilaton field ϕ has to satisfy, i.e. Eqs (5)-(7), follow from the variation principle of the action integral¹

$$S_{dilatons} = s_0 \int d^D x \sqrt{|g|} e^{-2\phi} \left(R - 4\nabla_a \phi \nabla^a \phi - \frac{1}{12} H^2 - \Lambda \right). \quad (9)$$

where the cosmological constant is $\Lambda = \frac{1}{a'} \frac{D-26}{3\pi^2}$, and s_0 is a rescaling constant. Since we are interested in cosmology, we can consider the antisymmetric parts vanishing, i.e. $H^2 = 0$. Therefore the action integral (9) takes the simpler form,

$$S_{dilatons} = s_0 \int d^D x \sqrt{|g|} e^{-2\phi} (R - 4\nabla_a \phi \nabla^a \phi - \Lambda), \quad (10)$$

which gives rise to an equivalent scalar-tensor model [7]

$$S_{ST} = \int d^D x \sqrt{|g|} [f(\phi) R - \omega(\phi) \nabla_a \phi \nabla^a \phi - V(\phi)] \quad (11)$$

with $f(\phi) = 4^{-1} \omega(\phi) = e^{-2\phi}$, and $V(\phi) = \Lambda f(\phi)$.

In the context of a D -dimensional spatially flat FLRW with line element

$$ds^2 = -dt^2 + a^2(t) \delta_{\alpha\beta} dx^\alpha dx^\beta, \quad (12)$$

it has been shown that there exist scale-factor duality. Specifically the action integral (9) and the field equations (5)-(7) are invariant under the discrete transformation² [25]

$$\bar{a} = a^{-1}, \quad \bar{\phi} = \phi - (D-1) \ln a, \quad (13)$$

which is of the form of Buscher's duality transformation, that is, the dilaton field admits a duality symmetry.

Consider now that the dimension of the FLRW spacetime is four, $D = 4$, that is, the Ricci scalar is given as follows,

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right]. \quad (14)$$

By replacing (14) in the action integral (10) the point Lagrangian is

$$L(a, \dot{a}, \phi, \dot{\phi}) = e^{-2\phi} (6a\dot{a}^2 - 12a^2\dot{a}\dot{\phi} + 4a^3\dot{\phi}^2 - \Lambda a^3), \quad (15)$$

where the field Eqs (5)-(7) can be written as

$$e^{-2\phi} (6a\dot{a} - 12a^2\dot{a}\dot{\phi} + 4a^3\dot{\phi}^2 - \Lambda) = 0, \quad (16)$$

¹ In the following we consider signature $(- + \dots)$.

² The scale-factor duality is a symmetry for (9) with or without the torsion term [25].

$$\ddot{a} + \frac{2}{a}\dot{a}^2 - 2\dot{a}\dot{\phi} = 0, \quad (17)$$

$$\ddot{\phi} + \frac{3}{2a^2}\dot{a}^2 - \dot{\phi}^2 - \frac{\Lambda}{4} = 0. \quad (18)$$

However it is not difficult to observe that the Brans-Dicke scalar field,

$$S_{BD} = \int d^D x \sqrt{|g|} \left(\psi R - \frac{\omega_{BD}}{\psi} \nabla_a \psi \nabla^a \psi - V_0 \psi \right) \quad (19)$$

with $\omega_{BD} = 1$, admits the scale-factor duality symmetry, while now the duality transformation is

$$\bar{a} = a^{-1}, \quad \bar{\psi} = \psi a^6. \quad (20)$$

The reason of this result is that the two action integrals (10) and (19) are related under the change of variables/fields, $\phi = -\frac{1}{2} \ln \psi$, when the Brans-Dicke parameter is $\omega_{BD} = 1$. Of course that property holds for all the scalar-tensor theories (11) where a new field Ψ can be defined, such as the action is (10). In that case the scale-factor duality symmetry exists, where the scalar field has to be shifted according to the formula connecting the field Ψ , and the dilaton field ϕ . Specifically all the scalar tensor models (11) with functions

$$f(\phi) = 4^{-1} e^{-2\Omega(\phi)}, \quad \omega(\phi) = (\Omega, \phi)^2 e^{-2\Omega(\phi)}, \quad V(\phi) = \Lambda f(\phi) \quad (21)$$

are invariant under a scale factor duality transformation (see also [43]).

Furthermore for the Brans-Dicke scalar field (19) it has been found that the field equations admits at least one Noether conservation law [26], with or without a dust fluid term. Because the conservation laws, and the corresponding symmetries, are independent of the “coordinate” systems, it means that the Lagrangian (15) of the dilaton field admits at least a Noether symmetry. The corresponding Noether point symmetry of Lagrangian (15) is given by the following vector field

$$X_N = a \partial_a + \frac{3}{2} \partial_\phi, \quad (22)$$

where the corresponding Noether conservation law is

$$I = 6a^2 e^{-2\phi} \dot{a}, \quad (23)$$

which is also invariant under the scale-factor duality transformation (13). This is not surprising since the form of the Lagrangian is the same, before or after the discrete transformation, but it is worthy to mention that either the sign of the conservation law does not change. Furthermore as we will see in Sec. (4), Lagrangians (15) and (19) are maximally symmetric which means that they admit eight Noether point symmetries as many as the free particle case in a two dimensional spacetime, or the two dimensional “oscillator”. However since the potential of the Lagrangian is not constant, as we will see in the following, this fact indicates that the classical equivalent dynamical system, which is described by (15) or (19), is the “oscillator” in the two dimensional flat space with Lorentzian signature³.

The application of Noether Symmetry Approach to gravitational theories is discussed in the following section. We show that for a class of cosmological models the existence of a Noether conservation law indicates the existence of a discrete transformation, i.e. discrete symmetry.

3. NOETHER SYMMETRIES AND DISCRETE TRANSFORMATIONS

Emmy Noether [27] proved that, under a point transformation with generator $X = \xi(t, x^k) \partial_t + \eta^i(t, x^k) \partial_i$, the action integral of a Lagrangian function $L(t, x^k, \dot{x}^k)$ transforms in a way such the following condition holds

$$X^{[1]} L + L \dot{\xi} = \dot{f} \quad (24)$$

³ We remark that the Noether symmetries are transformed as vector fields under coordinate transformations, that is, if J is the Jacobian of the coordinate transformation then, in the new coordinates, the symmetry vector X , is given as $\bar{X} = J(X)$.

then the Euler-Lagrange equations are invariants⁴, and X is called Noether (point) symmetry⁵. The second part of Noether's theorem states that if condition (24) holds for a given vector field X , then the following quantity is a conservation law for the Euler-Lagrange equations,

$$I = \xi \left(\dot{x}^k \frac{\partial L}{\partial \dot{x}^k} - L \right) - \eta^k \frac{\partial L}{\partial x^k} + f. \quad (25)$$

We remark that Noether's theorem extends for higher-order systems or for non-pointlike symmetries, however the above definition is the one that we are interested in this work.

Moreover, from the point symmetry X , the point transformation can be derived from the solution of the following system

$$t' = t + \varepsilon \xi(t, x^k) \quad , \quad x'^i = x^i + \varepsilon \eta^i(t, x^k), \quad (26)$$

where ε is an infinitesimal parameter of the point transformation.

We are interested on scalar field, mutiple-scalar field, scalar-tensor theories, and higher-order gravities such as, $f(R)$, $f(R, \square R, \dots)$, where the Lagrangian of the field equations can be written as follows, (either with the use of Lagrange multiplier)

$$L(x^k, \dot{x}^k) = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j - U(x^k), \quad (27)$$

where γ_{ij} , indicates the minisuperspace of the cosmological model, $i = \#_{spacetime}^{variables} + \#_{fields}^{variables}$, and $U(x^k)$ is the effective potential of the field equations, including the curvature components, which corresponds to the spacetime and the dynamic quantities of the new fields and matter terms when they are considered in the discussion. Furthermore, the gravitational field equations, which corresponds to the model with Lagrangian (27), are the set of the Hamiltonian function and the Euler-Lagrange equations, $E_L(L) = 0$. Finally, we have to point out that $L(x^k, \dot{x}^k)$ is autonomous. The Noether symmetry condition (24) for Lagrangians of the form (27) have been discussed in [21], and it has been found that the component η^i of the symmetry vector X is an element of the Homothetic algebra of γ_{ij} , while ξ is analogue to the Homothetic factor ψ of the homothetic vector field. As we have mentioned, Lagrangian (27) is autonomous, which means that the symmetry vector ∂_t exists and, at the same time, the field equations are invariant under time-reversal transformation, i.e. $t \rightarrow -t$ [25]. Following [21] and for the completeness of our analysis we consider the two cases $\dot{f} = 0$, and $\dot{f} \neq 0$ for the symmetry condition (24).

Let us consider $\dot{f} = 0$ in (24), then, for the Lagrangian (27), the Noether symmetry has the form, $X = 2\psi t \partial_t + H^i \partial_i$, and the following condition holds

$$\mathcal{L}_H U + 2\psi U = 0. \quad (28)$$

where H^i is the Homothetic vector of the tensor γ_{ij} , i.e. $\mathcal{L}_H \gamma_{ij} = 2\psi \gamma_{ij}$, and, when $\psi = 0$, H^i is an isometry (or a Killing vector). When H^i is an isometry then, from (28) and the Killing condition, we have that there exist a coordinate system $\{x^i\} \rightarrow \{y^I, y^A\}$, such as the isometry becomes ∂_I (these are called normal coordinates), and the Lagrangian (27) becomes, for a nonnull isometry [23],

$$L(y^k, \dot{y}^k) = \frac{1}{2} \left(\gamma_{II} (y^A) (\dot{y}^I)^2 + \gamma_{AB} (y^A) \dot{y}^A \dot{y}^B \right) - U(y^A), \quad (29)$$

where $A \neq I$ and $i = I, A$. However when the isometry is a null vector, i.e. $H^i H_i = 0$, in the normal coordinates, Lagrangian (27) becomes [29]

$$L(u, \dot{u}, v, \dot{v}, y^\beta, \dot{y}^\beta) = \frac{1}{2} (\gamma_{uu} (y^\beta, u) \dot{u}^2 + 2\gamma_{uv} (y^\beta, u) \dot{u} \dot{v} + \gamma_{\alpha\beta} (y^\beta, u) \dot{y}^\alpha \dot{y}^\beta) - U(y^\beta, u), \quad (30)$$

where the null isometry and the new coordinates are ∂_v and $\{x^i\} \rightarrow \{u, v, y^\beta\}$.

⁴ $X^{[1]} = X + (\dot{\eta}^i - \dot{x}^i \dot{\xi}) \partial_{\dot{x}^i}$ indicates the first prologation/extension of X in the space of variables $\{t, x^i, \dot{x}^i\}$ [28], and dot means derivative with respect to "t".

⁵ From (24) it is clear that it is not necessary the Lagrangian to be invariant under the point transformation. However a function \dot{f} can be introduced, where the latter is eliminated by the variational principle.

It is straightforward to see that, under the discrete “space” reversal transformation $y^I \rightarrow -y^I$, Lagrangian (29) is always invariant. On the other hand, this is not always true for (30): it is invariant under the reversal transformation $\{u, v\} \rightarrow \{-u, -v\}$ only for $\gamma_{ij}(y^\beta, -u) = \gamma_{ij}(y^\beta, u)$ and $U(y^\beta, -u) = U(y^\beta, u)$.

Now as far as concerns the original coordinate system, the reversal symmetry can be different and depends on the coordinate transformation. As we will see in the followings, the scale-factor duality transformation of the dilaton Lagrangian (15) is a reversal symmetry in the normal coordinates of the dynamical system.

On the other hand, when $\psi \neq 0$, from (28) and the Homothetic equation, we have that a coordinate system $\{x^i\} \rightarrow \{Y^I, Y^A\}$ exists where the Homothetic vector becomes $Y^I \partial_I$, while the Lagrangian (27) takes the following form

$$L(Y^k, \dot{Y}^k) = \frac{1}{2} \left(\gamma_{II} (Y^A) (\dot{Y}^I)^2 + (Y^I)^2 \gamma_{AB} (Y^A) \dot{Y}^A \dot{Y}^B \right) - \frac{\bar{U}(Y^A)}{(Y^I)^2}, \quad (31)$$

that is invariant under the reversal “coordinate” transformation $Y^I \rightarrow -Y^I$.

If $\dot{f} \neq 0$ in the symmetry condition (24), the generic Noether symmetry vector for (27) is $X = 2\psi \int T(t) \partial_t + T(t) S^i \partial_i$, when the following condition

$$\mathcal{L}_S U + 2\psi U + \sigma S = 0, \quad (32)$$

holds, where now S^i is a gradient Homothetic symmetry (or isometry) and function $T(t)$ is given by the second-order differential equation $T_{,tt} = \sigma T$.

In [30], the cases when $U(x^i)$ does not admit linear terms of the gradient function of the Killing isometry vector S_{KV} have been discussed. If S^i , in our notation, is a gradient (non-null) isometry vector field, in the normal coordinates $\{y^i\}$, Lagrangian (27) can be written with oscillatory terms

$$L(y^k, \dot{y}^k) = \frac{1}{2} \left((\dot{y}^I)^2 + \gamma_{AB} (y^A) \dot{y}^A \dot{y}^B \right) - \mu (y^I)^2 - \bar{U}(y^A), \quad (33)$$

Finally, if S^i is a gradient Homothetic symmetry in the normal coordinates $\{Y^i\}$, Lagrangian (27) describes the generalized Riemannian Ermakov-Pinney system

$$L(Y^k, \dot{Y}^k) = \frac{1}{2} \left((\dot{Y}^I)^2 + (Y^I)^2 \gamma_{AB} (Y^A) \dot{Y}^A \dot{Y}^B \right) - \mu (Y^I)^2 - \frac{\bar{U}(Y^A)}{(Y^I)^2}. \quad (34)$$

Therefore Lagrangians (33), (34) are invariant under the discrete transformations $y^I \rightarrow -y^I$ and $Y^I \rightarrow -Y^I$. Furthermore, we observe that if $\mu = 0$, and $\gamma_{II} = 1$, then Lagrangians (33), (34) are equivalent to Lagrangians (29) and (31) respectively.

Now, if the minisuperspace γ_{ij} admits a gradient isometry and the potential $U(x^i)$ admits linear terms of S_{KV} , then, in the normal coordinates $\{y^i\}$, it is $S_{KV} = y^I$, and, in the normal coordinates, the effective potential is $U(y^i) = \sigma y^I + \bar{U}(y^A)$, $\sigma \neq 0$. In this case a Noether conservation laws exists and, however, the reversal symmetry does not exist.

Of course, the above analysis holds for a single symmetry vector acting on the Lagrangian. If more than one symmetries exist, then new discrete transformations can be found. However it is worth stressing that, while in one coordinate system the discrete transformation can be seen as a reversal transformation, in another coordinate system, it can be a different transformation.

In the case where the Noether symmetry is generated by a null isometry, i.e. the Lagrangian has the form of (30) for $\gamma_{uu} = 0$, then, under the transformation $\{u \rightarrow v, v \rightarrow u\}$, the dynamical system is not necessary invariant. However the two dynamical systems are related by a simple change of variables. When the elements of the metric γ_{ij} , i.e. $\gamma_{uv}(y^\nu, u)$, $\gamma_{ij}(y^\nu, u)$, and the potential $\bar{U}(y^\beta, u)$ are not dependent on the variable u , then the dynamical system is invariant. If that is true, then it is easy to show that the dynamical system admits more than one Noether conservation laws.

In the following section, we apply the above results to some cosmological models where Noether point symmetries can be identified.

4. DUALITY TRANSFORMATIONS FROM NOETHER SYMMETRIES

Let us consider a simple well-known system in \mathcal{M}^2 space which follows from the Lagrangian

$$L(x, \dot{x}, y, \dot{y}) = \frac{\Omega}{2} (\dot{x}^2 - \dot{y}^2) - \frac{\mu}{2} (x^2 - y^2), \quad (35)$$

where Ω, μ are constants. Lagrangian (35) describes the two-dimensional “oscillator” in a flat space with Lorentzian signature. It is straightforward to see that (35) admits eight Noether point symmetries (the Noether symmetries can be found for instance in [31, 32]). Also the dynamical system in which Lagrangian (35) describes is invariant under the following discrete transformations $\{x \rightarrow -x\}$, or $\{y \rightarrow -y\}$, and the complex one $\{x \rightarrow iy, y \rightarrow ix\}$, while, under the transformation $\{x \rightarrow y, y \rightarrow x\}$, the new Lagrangian \bar{L} , is $\bar{L} = -L$. However while the field equations have same the sign, the Hamiltonian constant changes: if the latter is zero then everything is invariant. As we can see Lagrangian (35) is of the form of Lagrangian (33). Hence these discrete symmetries for Lagrangian (35) follows directly from the existence of the Noether symmetry vectors as we discussed in Sec. 3.

Specifically the transformations $\{x \rightarrow -x\}$, $\{y \rightarrow -y\}$ follow from the translation symmetries while the remaining two transformations follow from the rotation symmetry and the homothetic symmetry of M^2 space. As we will show below, these discrete symmetries while are reversal symmetries in the Cartesian coordinates, in the coordinates where the dilaton field is defined, take the form of the scale factor duality symmetry.

Consider now the coordinate transformation,

$$x = \frac{\sqrt{2}}{2} (u + v), \quad y = \frac{\sqrt{2}}{2} (u - v), \quad (36)$$

with inverse

$$u = \frac{\sqrt{2}}{2} (x + y), \quad v = \frac{\sqrt{2}}{2} (x - y). \quad (37)$$

Hence, the discrete transformation $\{x \rightarrow -x, y \rightarrow -y\}$ in the new coordinates system corresponds to $\{u \rightarrow -u, v \rightarrow -v\}$, while the transformation $\{x \rightarrow -x\}$ corresponds to $\{u \rightarrow -v, v \rightarrow -u\}$, and $\{y \rightarrow -y\}$ corresponds to the discrete transformation $\{u \rightarrow v, v \rightarrow u\}$. All these transformations are related with the admitted translation group of the flat space.

In the new coordinate system $\{u, v\}$, Lagrangian (35) becomes

$$L(u, \dot{u}, v, \dot{v}) = \Omega(\dot{u}\dot{v}) - \mu(uv). \quad (38)$$

We assume the second coordinate transformation

$$a = u^{p^+} v^{p^-}, \quad \phi = \ln(u^{q^+} v^{q^-}), \quad (39)$$

in which the constants p_{\pm}, q_{\pm} are

$$p_{\pm} = \frac{(\omega - 4) \sqrt{6(6 - \omega)} \pm 4(\omega - 6)}{\sqrt{6(6 - \omega)}(3\omega - 16)}, \quad (40)$$

$$q_{\pm} = 2 \frac{\sqrt{6(6 - \omega)} \pm 3(\omega - 6)}{\sqrt{6(6 - \omega)}(3\omega - 16)}. \quad (41)$$

where

$$\Omega = 8 \frac{(\omega - 6)}{3\omega - 16}. \quad (42)$$

Then under the second transformation, Lagrangian (38) becomes

$$L(a, \dot{a}, \phi, \dot{\phi}) = e^{-2\phi} (6a\dot{a}^2 - 12a^2\dot{a}\dot{\phi} + \omega a^3\dot{\phi}^2 - \Lambda a^3), \quad (43)$$

where the constant $\Lambda = \mu$. The latter Lagrangian has the form of (15) with a difference in the coefficient of $\dot{\phi}^2$. Furthermore, it is easy to see that, under the transformation $\phi = -\frac{1}{2} \ln(\psi)$, Lagrangian (43) becomes of Brans-Dicke form as (19) where $\omega = 4\omega_{BD}$.

From transformations (39), we see that the discrete transformation $\{y \rightarrow -y\}$, in the Cartesian coordinates, or $\{u \rightarrow v, v \rightarrow u\}$ in the coordinates $\{u, v\}$, is that of the scale-factor duality

$$a \rightarrow a^{-1} \quad (44)$$

if and only if $p_+ = -p_-$. Using (40), we find the unique solution $\omega = 4$, or $\omega_{BD} = 1$, where (41) corresponds to the Lagrangian of the dilaton field (15). Therefore we conclude that the scale-factor duality transformation is related to the existence of Noether symmetries of the Lagrangian and the field equations. The reason why only the constant $\omega_{BD} = 1$ is admitted for the a scale-factor duality follows from the property of the coordinate transformation (39).

In general, from (39), we can achieve the inverse transformation

$$u = a^{Q_-} \exp(-P_- \phi), \quad v = a^{-Q_+} \exp(P_+ \phi), \quad (45)$$

where the new constants are

$$P_{\pm} = \frac{p_{\pm}}{p_+ q_- - p_- q_+}, \quad Q_{\pm} = \frac{q_{\pm}}{p_+ q_- - p_- q_+}, \quad (46)$$

that is, it is possible to construct discrete transformations where the scalar-tensor model (43) is invariant under Noether symmetries.

Hence, by using (39) and (45), we have that the discrete transformation $\{u \rightarrow v, v \rightarrow u\}$ in the non-diagonal coordinates $\{u, v\}$, the coordinates $\{a, \phi\}$ become

$$a \rightarrow a^{(p_- Q_- - p_+ Q_+)} \exp((p_+ P_+ - p_- P_-) \phi), \quad (47)$$

$$\exp(\phi) \rightarrow a^{(q_- Q_- - q_+ Q_+)} \exp((q_+ P_+ - q_- P_-) \phi), \quad (48)$$

and Lagrangian (43) is invariant. Transformation (47), (48) is a discrete symmetry for (43). Furthermore, as far as concerns the Brans-Dicke field (19), the discrete transformation which does not change the field equations is

$$a \rightarrow a^{(p_- Q_- - p_+ Q_+)} \psi^{-\frac{1}{2}(p_+ P_+ - p_- P_-)}, \quad (49)$$

$$\psi \rightarrow a^{-2(q_- Q_- - q_+ Q_+)} \psi^{(q_+ P_+ - q_- P_-)}. \quad (50)$$

In order to clarify differences between discrete and local transformations, let us consider the Noether symmetry vector (22) for the Brans-Dicke scalar field (19). Then, from (26), we find the point transformation

$$t \rightarrow t, \quad a \rightarrow e^{\varepsilon} a, \quad \phi \rightarrow \phi + \frac{3}{2} \varepsilon, \quad (51)$$

which means that a rescaling of the scale factor keeps invariant, not only the Lagrangian, but also the action integral if the field ϕ shifts properly. Of course, for the remaining Noether symmetries of the model, new point transformations can be constructed. However, while from a Noether symmetry we can always construct a continuous transformation from the system (26) in order to find the discrete transformation, which leaves the field equations invariant, we have to write the latter in the normal coordinates of the symmetry vector, where there the discrete symmetry is a reversal symmetry.

Furthermore, in order the reversal symmetry to provide a discrete symmetry in the original coordinates of the dynamical system, the reversal transformation should survive. To demonstrate this fact, let us consider the ‘‘polar’’ coordinates of (35) and let $\Omega = 1$, it is

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2} (\dot{r}^2 - r^2 \dot{\theta}^2) - \frac{\mu}{2} r^2, \quad (52)$$

from where we can see that the latter Lagrangian is invariant under the discrete transformations $\{r \rightarrow -r\}$, $\{\theta \rightarrow -\theta\}$, while $\theta \rightarrow -\theta$, is a transformation which follows from the $\{x \rightarrow -x\}$ or $\{y \rightarrow -y\}$. If we assume the second coordinate transformation $r^2 = \sqrt{\frac{8}{3}} a^3$, $\theta = \sqrt{\frac{3}{8}} \Phi$, then (52) becomes

$$L(a, \dot{a}, \Phi, \dot{\Phi}) = 3a\dot{a}^2 - \frac{1}{2} a^3 \dot{\Phi}^2 + a^3 \Lambda \quad (53)$$

where $\mu = -2\sqrt{\frac{3}{8}}\Lambda$. From this fact, we can see that, from the above two discrete transformations, only the $\{\theta \rightarrow -\theta\}$, survives and becomes $\{\Phi \rightarrow -\Phi\}$. Of course Lagrangian (53) is that of the minimally coupled scalar field with constant potential. Finally, if we assume that $\Psi = \exp(\Phi)$, then the discrete transformation $\{\Phi \rightarrow -\Phi\}$ becomes a duality symmetry on the minisuperspace $\{\Psi \rightarrow \Psi^{-1}\}$. Let us remark that the latter symmetry for (53) and the scale factor duality symmetry for Lagrangian (43) are related by the reversal symmetry $\{y \rightarrow -y\}$ of (35) in the Cartesian coordinates.

The admitted common Noether algebra for the Brans-Dicke field (43), and the scalar field (53) is not surprising [48] since the two Lagrangians are related under a conformal transformation which relates the Jordan and the Einstein frames [49].

4.1. Duality transformation in $f(R)$ -gravity

In the above dilaton field Lagrangian, we studied the scalar-factor duality. However, in order to search for discrete transformations which relates different models, let us consider a spatially flat FLRW spacetime with line element

$$ds^2 = -\frac{1}{N^2(t)}dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (54)$$

Starting from the O'Hanlon theory [33], the equivalent $f(R)$ -gravity, in the metric formalism can be easily achieved [34]. Adopting a Lagrange multiplier as in [35], the above point-like Lagrangian can be suitably recast as

$$L(N, a, \dot{a}, \dot{\phi}, \dot{\phi}) = N(a, \phi) \left(6a\dot{\phi}\dot{a}^2 + 6a\dot{a}\dot{\phi} \right) + \frac{a^3 V(\phi)}{N(a, \phi)}, \quad (55)$$

where we have considered $N(t) = N(a(t), \phi(t))$. Furthermore, the relation between the O'Hanlon scalar field and $f(R)$ gravity is

$$\phi = f'(R) \quad , \quad V(\phi) = f'(R)R - f, \quad (56)$$

where the latter first order differential equation is a Clairaut differential equation which, for specific potentials $V(\phi)$, provides the functional form of $f(R)$.

Let us assume now that $N(a, \phi) = a^{2\lambda-1}\phi^\lambda$. The minisuperspace of (55) admits the maximal dimensional homothetic algebra. Hence, for potential of the form

$$V(\phi) = V_0 \phi^\lambda (\phi^{\lambda+1} + 1)^{-\frac{2(\lambda-2)}{\lambda+1}}, \quad \text{for } \lambda \neq -1 \quad (57)$$

Lagrangian (56) admits conservation laws which follow from the nonnull isometry vector

$$X_{f(R)} = a^{-\lambda} \partial_a - a^{-1-\lambda} \phi^{-\lambda} (1 + \phi^{1-\lambda}) \partial_\phi. \quad (58)$$

Under the coordinate transformation

$$a = u^{\frac{1}{\lambda+1}}, \quad \phi = (uv^{-1})^{\frac{1}{\lambda+1}}, \quad (59)$$

the Lagrangian (55) becomes

$$L(u, \dot{u}, v, \dot{v}) = \frac{6}{(\lambda+1)^2} \dot{u}\dot{v} + V_0 (u+v)^{-2(\frac{\lambda-2}{\lambda+1})}. \quad (60)$$

It is not necessary to go in Cartesian coordinates to see that (60) is invariant under the discrete transformation $\{u \rightarrow v, v \rightarrow u\}$. Therefore, from (59), we can see that the Lagrangian is invariant under the transformation

$$a \rightarrow a, \quad \phi \rightarrow \phi^{-1}.$$

We shall say that the solution of the field equations among the two spacetimes

$$ds^2 = -\frac{1}{a^{2\lambda-1}\phi^\lambda} dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (61)$$

$$ds^2 = -\frac{\psi^\lambda}{a^{2\lambda-1}} dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (62)$$

in the O'Hanlon theory, or in $f(R)$ gravity, are related under the transformation $\phi = \psi^{-1}$, when the potentials of the two theories $V(\psi)$, $V(\phi)$ are related as $V(\phi) = V(\psi^{-1})$, where $V(\phi)$ is that of (57). However, these two models provide the same scale-factor but in different lapse time. Which means that in the lapse time $d\tau = \frac{1}{N}dt$, the two theories provide different scale factors, i.e. cosmological solutions.

On the other hand, this property is useful for theories where the lapse time is energy dependent, as in the so-called Gravity's Rainbow [36]. In other words, lapse time can be considered to be variable in terms of the dynamical quantities of the model. See, for instance, [37, 38].

Furthermore, from (56) and for $\lambda = 2$, we can find a closed-form for $f(R)$. Hence, $f(R) \simeq R^2$, which is a power law $f(R)$ model which provides a de Sitter universe [39]. For $\lambda = 0$, from (56), we have the model

$$f(R) \simeq -R - f_1 R^{\frac{4}{3}}, \quad (63)$$

where the lapse functions in the two metrics (61) and (62) are $N_1(a, \phi) = N_2(a, \psi) = a$. It is interesting to mention that this model has been found previously from the application of the Killing tensors in the Lagrangian of the field equations [40], and it has been found that describes the Chevallier, Polarski and Linder (CPL) parametric dark energy model.

The results presented here are different from those of previous studies on the relation between duality and Noether symmetries, and, in some sense, generalize them [41–43]. Specifically, if we consider the calculations in [41] and assume the present approach, we will see that what is there considered a conserved quantity, is conserved if and only if the $f(R)$ function is going to vanish. This fact follows from the complete solution of the Noether symmetry condition adopted here. On the other hand, using the specific approach for Noether symmetries adopted in [41] for Lagrangian (55) with $N = 1$, only the $f(R) \simeq R^{3/2}$ satisfies the Noether symmetry condition [44, 45]. The problem can be bypassed assuming a generic $f(R)$ gravity model and changing the point-like Lagrangian which is no more that of $f(R)$ gravity derived in the metric formalism. This can be easily seen by comparing the power-law solution found for $f(R) = \sqrt{R}$ in [41], with the power law solution which follows from (55) for the same functional form of $f(R)$ [46]. Furthermore, by using the classification for the Noether symmetries in $f(R)$ gravity for the lapse time $N = 1$, as in [46], we can see that the scale factor duality, which follows from Noether point symmetries, does not exist in $f(R)$ gravity. As a final comment, we have to say that the main role in determining the symmetries for $f(R)$ gravity is played by the lapse function N and the form of the point-like Lagrangian adopted⁶. The above results are general and results in [41] can be easily framed in this approach being careful with the above remarks.

5. CONCLUSIONS

The relation between discrete transformations and Noether (point) symmetries for cosmological models in the context of the minisuperspace has been discussed. Using results from differential geometry, Noether (point) symmetries are generated by the elements of the Homothetic algebra of the given minisuperspace. We discussed cosmological dynamical systems that admit a Noether symmetry in the normal coordinates of the Noether symmetry vector. Furthermore, in the normal coordinates, we studied the existence of discrete transformations. We applied that results and we showed that Brans-Dicke-like cosmological models with linear potentials admit a discrete transformation which keeps the field equations invariant, while when the Brans-Dicke parameter is $\omega_{BD} = 1$, the discrete transformation is a scale-factor duality transformation and the Brans-Dicke scalar field is equivalent with the dilaton scalar field. Finally, for arbitrary Brans-Dicke parameter, a discrete transformation which leaves invariant the field equations has been found.

Furthermore we showed that there exist a family of scalar-field models, equivalent in O’Hanlon gravity and in $f(R)$ -gravity, where the solutions of the models are related under discrete transformations for the scalar field. A particular transformation is a duality transformation on the minisuperspace.

In particular what we showed is that the existence of a Noether symmetry for the gravitational Lagrangian can provide discrete symmetries for the field equations. These discrete symmetries are reversal transformations in the normal coordinates of the symmetry vector, while in other coordinates, when they survive, can give another behavior for the original system.

Indeed the Brans-Dicke field with linear potential, consequently the dilaton field, and the minimally coupled scalar field with constant potential are maximally symmetric and describe the same classical mechanical system of the “oscillator” in the two dimensional flat space with Lorentzian signature. However, while the reversal symmetry $\{y \rightarrow -y\}$, for the “oscillator”, becomes a scale-factor duality symmetry for the dilaton field, or more general, the discrete symmetry (49)-(50) for the Brans-Dicke field for the minimally coupled field, the reversal symmetry is again a reversal symmetry for the scalar field.

An issue that we did not discussed in details on that work is what happens as far as concerns the scale-factor duality transformation under conformal transformations. As we mentioned above in the case of the dilaton field with action integral (10) defined in the Jordan frame, in the Einstein frame is equivalent to that of a minimally coupled scalar field with constant potentials, where, obviously, the field equations does not admit scale-factor duality symmetry but

⁶ For a discussion see also [47].

the latter becomes a reversal symmetry for the scalar field. The reason for that is that Noether symmetries survive under conformal transformations [47–49]. Hence, in order to explore the relation between point transformations and discrete transformations in conformal equivalent theories, in a forthcoming work we will extend this analysis to the Lie point symmetries of the Wheeler-DeWitt equation.

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